SECTION 4.2: THE MEAN VALUE THEOREM

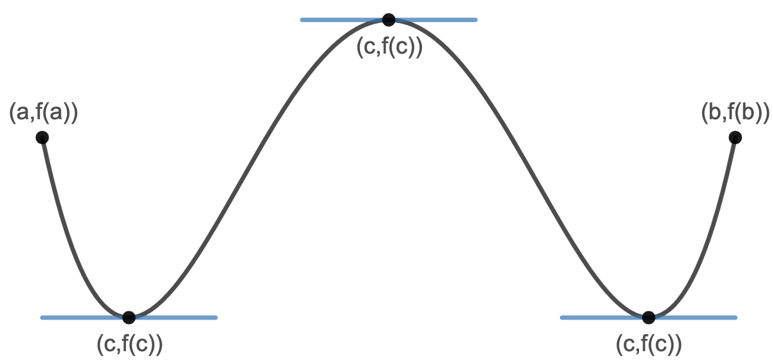
This section aims to develop one of the most important theorems in Calculus, the Mean Value Theorem.
We begin with (what appears to be) a more specialized result, Rolle's Theorem.

ROLLE'S THEOREM: Suppose a function f satisfies the following conditions:

1. f is **continuous** on $[a, b]$
2. f is **differentiable** on (a, b)
3. $f(a) = f(b)$.

Then there is at least one value c in (a, b) with $f'(c) = 0$.

Geometrically, we can imagine the situation sketched out below.



PROOF:

To prove Rolle's Theorem, we assume f is continuous on $[a, b]$, f is differentiable on (a, b), and that $f(a) = f(b)$.
If f is constant, then $f'(x) = 0$ for all x in (a, b), so the theorem is true in this case.
If f isn't constant, then since f is continuous on $[a, b]$, the EVT guarantees f attains both its absolute maximum and absolute minimum on $[a, b]$.
We know these extrema could occur at the endpoints or at local extrema. Since $f(a) = f(b)$ and we're assuming f isn't constant, at least one of these extreme values occurs at local extremum $(c, f(c))$ for some c in (a, b).
Since f is differentiable on (a, b), Fermat's Theorem guarantees $f'(c) = 0$ at this point, so the theorem is true in this case as well.

EXAMPLE 1: Let $f(x) = -x^3 + 6x^2 + 2$.

1. Show f satisfies the conditions for Rolle's Theorem on $[0, 6]$.

   f is a polynomial so f is continuous on $[0, 6]$.

   $f'(x) = -3x^2 + 12x$ exists for all x in $(0, 6)$.

   $f(0) = -(0)^3 + 6(0)^2 + 2 = 2$ and $f(6) = -(6)^3 + 6(6)^2 + 2 = 2$ so $f(0) = f(6)$.

2. Find all values c guaranteed by Rolle's Theorem in $(0, 6)$.

   We solve $f'(x) = -3x^2 + 12x = 0$. We get $-3x(x - 4) = 0$ so $x = 0$ or $x = 4$.

   Only $x = 4$ lies in the interval $(0, 6)$ so $x = 4$ is the value guaranteed by Rolle's Theorem.

EXAMPLE 2: The function $f(x) = |x|$ is continuous on $[-2, 2]$ and satisfies $f(-2) = 2 = f(2)$.
Yet there is nowhere in the interval $(-2, 2)$ with $f'(x) = 0$. Why does this not contradict Rolle's Theorem?
f is not differentiable at $x = 0$ owing to the corner at $(0, 0)$. Hence, Rolle's Theorem doesn't apply.

EXAMPLE 3: What goes up... must come down. How does Rolle's Theorem prove that if an object is thrown into the air then at some point it must be at rest before it returns to the ground?

We are now ready to entertain the main feature of the section:

THE MEAN VALUE THEOREM (MVT): Suppose a function f satisfies the following conditions:

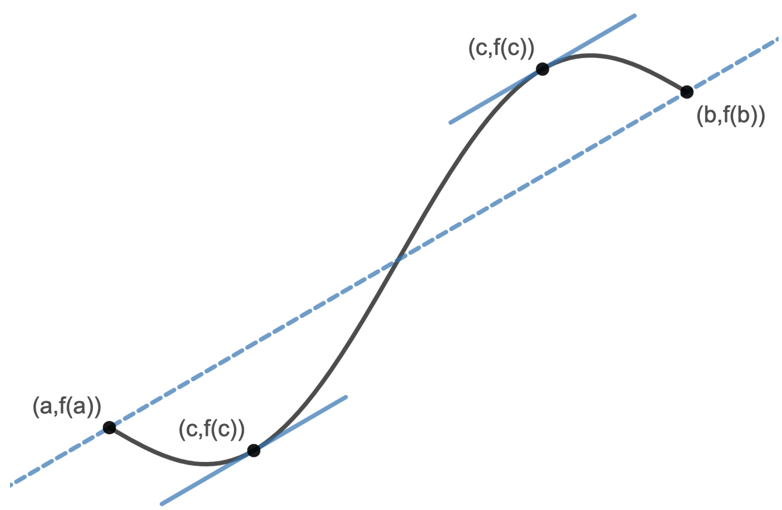
1. f is **continuous** on $[a, b]$
2. f is **differentiable** on (a, b)

Then there is at least one value c in (a, b) which satisies: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

As usual, it helps to visualize this theorem. Recall that $f'(c)$ is the slope of the tangent line at $(c, f(c))$.

The quantity $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.

For these two slopes to be equal means the tangent line at some point is parallel to the secant line, as seen below.



In terms of **rates of change**, the MVT says that at some point in the interval (a, b), **instantaneous** rate of change of f **at a point** equals the **average** rate of change of f over **the interval** $[a, b]$.

EXAMPLE 4: Let $f(x) = \cos(2x) - x$.

1. Show f satisfies the conditions for The Mean Value Theorem on $[0, \pi]$.

   f is a difference of two continuous functions, $\cos(2x)$ and x, so f is continuous on $[0, \pi]$.

   $f'(x) = -2\sin(2x) - 1$ exists for all x in $(0, \pi)$.

2. Find all values c guaranteed by the Mean Value Theorem in $(0, \pi)$.

   We first find the average rate of change: $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{(1 - \pi) - 1}{\pi - 0} = \dfrac{-\pi}{\pi} = -1.$

   Next, we solve $f'(x) = -2\sin(2x) - 1 = -1$. We get $\sin(2x) = 0$ so $2x = \pi k$ for $k = 0, \pm 1, \pm 2, \ldots$.

   We get $x = \frac{\pi}{2} k$ for $k = 0, \pm 1, \pm 2, \ldots$, and of these only $x = \frac{\pi}{2}$ lies in the interval $(0, \pi)$.

NOTE: It is worth checking this example using desmos.
Graph the function, the secant line, and the tangent line to see the relationship between them.

EXAMPLE 5: Suppose you drive 150 miles in 2 hours.

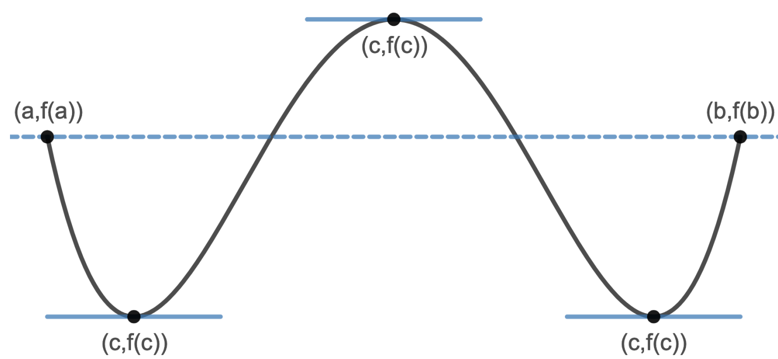
Use the MVT to prove that at some point you were traveling at exactly 75 miles per hour.

EXAMPLE 6: Show the MVT reduces to Rolle's Theorem in the case $f(a) = f(b)$.

For Rolle's Theorem, we assume f is continuous on $[a, b]$, f is differentiable on (a, b), and $f(a) = f(b)$.

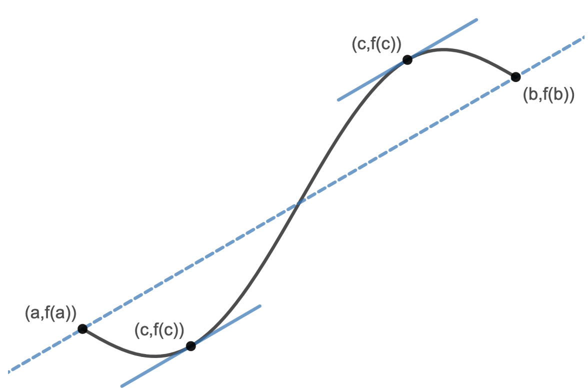
In this case, $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f(a) - f(a)}{b - a} = 0$. Hence, the MVT guarantees values c with $f'(c) = 0$.

Hence, we can view Rolle's Theorem as just a special case of the MVT. Geometrically:

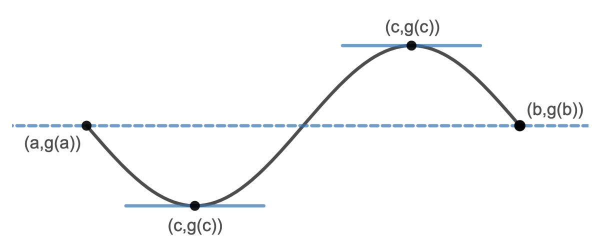


Even though Rolle's Theorem is a special case of the MVT, it turns out the only need thing we need to prove the MVT is Rolle's Theorem. This means in the grand scheme of things, the MVT and Rolle's Theorem are mathematically equivalent - they are expressing precisely the same mathematical sentiment.

PROOF OF THE MVT: The idea behind the proof of the MVT is to assume f satisfies the conditions of the MVT then reduce the situation back to a Rolle's Theorem problem. The way we do this is to consider a function $g(x)$ which is defined as $g(x) = f(x) - [\text{the secant line through } (a, f(a)) \text{ and } (b, f(b))]$. Geometrically, we are taking the problem below on the left to the problem below on the right.



$y = f(x)$ with secant line (dashed)



$y = g(x) = f(x) - [\text{the secant line}]$

More specifically, we define $g(x) = f(x) - \left[\left(\dfrac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right]$.

We now show g satisfies the conditions of Rolle's Theorem:

1. g is the difference between f (a continuous function by assumption) and a linear function so g is continuous.

2. $g'(x) = D_x \left\{ f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right] \right\} = f'(x) - \left(\frac{f(b) - f(a)}{b - a} \right)$ exists for all x in (a, b).

3. We need to show $g(a) = g(b)$:

   $$g(a) = f(a) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (a - a) + f(a) \right] = f(a) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (0) + f(a) \right] = f(a) - f(a) = 0.$$

   $$g(b) = f(a) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (b - a) + f(a) \right] = f(a) - [f(b) - f(a) + f(a)] = f(b) - f(b) = 0.$$

   NOTE: Since g is the difference between f and the secant line through $(a, f(a))$ and $(b, f(b))$, it should be no surprise that $g(a) = g(b) = 0$.

By Rolle's Theorem, we are guaranteed at least one value c in the interval (a, b) with $g'(c) = 0$.

Since $g'(x) = f'(x) - \left(\frac{f(b) - f(a)}{b - a} \right)$, $g'(c) = 0$ means $f'(c) - \left(\frac{f(b) - f(a)}{b - a} \right) = 0$ or $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Hence, we have proven the Mean Value Theorem.

We'll see how the MVT becomes the mathematical MVP in the sections and chapters that follow!

HOMEWORK: Section 4.2: 1 - 49 every other odd.